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INSTABILITY IN THE FIRST APPROXIMATION FOR TIME-DEPENDENT LINEARIZATIONS†

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The concept of Krasovskii stability is introduced. The criterion of Krasovskii instability for time-dependent linearizations is obtained. This criterion is compared with Chetayev's theorem.

The theory of stability in the first approximation for time-dependent aperiodic linearizations has now been well developed [1–9]. Many methods of investigation and results on different types of stability: Lyapunov stability, asymptotic stability and exponential stability, are described in these references.

The situation is quite different with instability criteria. Here, only one result is widely known: Chetayev's theorem on instability in the first approximation [10, 6].

It must be pointed out that interest in problems of instability has been stimulated by the study of strange attractors and the chaotic dynamics of non-linear systems [11–14]. As a rule, linearization along most trajectories belonging to a strange attractor is time-dependent and aperiodic and has a positive Lyapunov exponent.

Another important sphere of application of instability criteria is control theory, where use is often made of time-dependent inverse relations [15, 16]. Here, non-linearities are not always differentiable. Therefore, only requirements of continuity in the neighbourhood of the trivial solution will be imposed here.

Consider the system

$$dx/dt = F(t,x), \quad t \geqslant 0, \quad x \in \mathbb{R}^n \tag{1}$$

where $F(t, x)$ is a continuous vector function such that $F(t, 0) \equiv 0$.

By slightly extending the concept of exponential stability [7, 8] for the case of a zero exponent, we will introduce the following definition.

Definition. The trivial solution $x(t) \equiv 0$ of system (1) will be termed Krasovskii stable if numbers R and ε exist such that, for any solution $x(t, t_0, x_0)$ with the initial data $|x_0| \leqslant \varepsilon$, the following criterion is satisfied

$$|x(t,t_0,x_0)| \leqslant R|x_0|, \quad \forall t \geqslant t_0 \tag{2}$$

where $|x|$ is the Euclidean norm of the vector x, $x(t_0, t_0, x_0) = x_0$.

Note that the number R does not depend on the choice of the vector x_0 from the sphere $\{|x| \leqslant \varepsilon\}$. If R is independent of the choice of $t_0 \in [0, +\infty]$, we will speak of uniform Krasovskii stability.

The obvious relations between property (2) and Lyapunov stability have been discussed in [4, p. 23]. Note that many classical criteria of Lyapunov stability [1–9] are also criteria of Krasovskii stability.

We will examine the system

$$dx/dt = A(t)x + f(t,x), \quad t \geqslant 0, \quad x \in \mathbb{R}^n \tag{3}$$

where $A(t)$ is a continuous $n \times n$ matrix bounded in $[0, +\infty)$. We will assume that the vector function $f(t, x)$ is continuous, and that in a certain neighbourhood $U(0)$ of the point $x = 0$ the following inequality is satisfied

$$|f(t,x)| \leqslant \varkappa|x|^{\nu}, \quad \forall t \geqslant 0, \ \forall x \in U(0); \quad \varkappa > 0, \ \nu > 1 \tag{4}$$

We will introduce into consideration the fundamental matrix $Z(t) = (z_1(t), \ldots, z_n(t))$ composed of linearly independent solutions $z_j(t)$ of the system

$$dz/dt = A(t)z \tag{5}$$

Theorem. If the inequality

$$\sup_k \varliminf_{t \to +\infty} \left[\frac{1}{t}\left(I(t) - \sum_{j \neq k} \ln|z_j(t)|\right)\right] > 0, \quad I(t) = \int_0^t \operatorname{tr} A(\tau)d\tau \tag{6}$$

is satisfied, then the solution $x(t) \equiv 0$ of system (3) is Krasovskii unstable.

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Proof. We will assume that $t_0 = 0$ and carry out Schmidt's orthogonalization procedure

$$v_1(t) = z_1(t), \quad v_2(t) = z_2(t) - v_1(t)^* z_2(t)\frac{v_1(t)}{|v_1(t)|^2}, \ \ldots$$

$$\ldots, v_n(t) = z_n(t) - v_1(t)^* z_n(t)\frac{v_1(t)}{|v_1(t)|^2} - \ldots - v_{n-1}(t)^* z_n(t)\frac{v_{n-1}(t)}{|v_{n-1}(t)|^2}$$

These relations lead to the equations

$$v_i(t)^* v_j(t) = 0, \quad \forall j \neq i; \quad |v_j(t)|^2 = v_j(t)^* z_j(t)$$

The latter equation implies the criterion

$$|v_j(t)| \leqslant |z_j(t)| \tag{7}$$

We will introduce into consideration the unitary matrix

$$U(t) = \left(\frac{v_1(t)}{|v_1(t)|}, \ \ldots, \frac{v_n(t)}{|v_n(t)|} \right)$$

and make the replacement $x = U(t)y$ in system (3)

$$dy/dt = B(t)y + g(t, y) \tag{8}$$

where

$$B(t) = U(t)^{-1} A(t)U(t) - U(t)^{-1} \dot{U}(t), \quad g(t,y) = U(t)^{-1} f(t, U(t)y)$$

It is well known [8] that $B(t)$ is an upper triangular matrix with diagonal elements $b_{jj}(t)$ satisfying the condition

$$J_{jj}(t) \equiv \int_0^t b_{jj}(\tau)d\tau = \ln\frac{|v_j(t)|}{|v_j(0)|} \tag{9}$$

Note that, by virtue of the orthogonalization process, the following relation is satisfied

$$Z(t) = U(t)Q(t)$$

where $Q(t)$ is an upper triangular matrix with diagonal elements $|v_j(t)|$. Therefore, the following identity holds

$$\prod_{j=1}^{n} |v_j(t)| = |\det Z(t)| = |\det Z(0)| \exp I(t)$$

From this and relations (7) and (9), we obtain the criterion

$$J_{jj}(t) \geqslant \ln\frac{|\det Z(0)| \exp I(t)}{\prod\limits_{i\neq j} |z_i(t)||z_j(0)|} = I(t) - \sum_{i \neq j} \ln|z_i(t)| - \ln|z_j(0)| + \ln|\det Z(0)| \tag{10}$$

Maintaining generality, in condition (6) it is possible to consider that a supremum is reached when $k = n$. Therefore, relations (6) and (10) indicate the existence of a number $\mu > 0$ such that, for sufficiently large t, the criterion

$$J_{nn}(t) \geqslant \mu t \tag{11}$$

is satisfied. From the last equation of system (8)

$$\dot{y}_n = b_{nn}(t)y_n + g_n(t,y)$$

we obtain the equation

$$y_n(t) = \exp J_{nn}(t)\left(y_n(0) + \int_0^t (\exp(-J_{nn}(\tau))g_n(\tau, y(\tau))d\tau \right) \tag{12}$$

Assuming that criterion (2) is satisfied, from condition (4) and orthogonality $U(t)$ we obtain the relation

$$|g(t,y(t))| \leq \varkappa R^{\nu} |y(0)|^{\nu} \quad (13)$$

Note that criterion (11) indicates the existence of a number p for which the following inequality is satisfied

$$\int_0^t \exp(-J_{nn}(\tau))d\tau \leq \rho \quad \forall t \geq 0 \quad (14)$$

We will now select the initial data $y(0)$ such that $y_n(0) = |y(0)| = \delta$ and the number δ satisfies the relations

$$\delta \leq \varepsilon, \quad \delta > \rho\varkappa R^{\nu}\delta^{\nu} \quad (15)$$

The existence of a number δ satisfying the second inequality of (15) follows from the condition $\nu > 1$.
From formula (12), criterion (11) and inequalities (13)–(15) we obtain the relation

$$\lim_{t \to +\infty} y_n(t) = +\infty$$

This contradicts the assumption that inequality (2) is satisfied. Consequently, the solution $x(t) \equiv 0$ is Krasovskii unstable. The theorem is proved.

Remarks. 1. In fact, a slightly stronger result has been proved: if condition (6) is satisfied, the following criterion cannot occur

$$|x(t,t_0,x_0)| \leq R|x_0|^{\alpha}, \quad \forall t \geq t_0$$

where α is any positive number satisfying the inequality

$$\alpha > \nu^{-1}$$

2. Conditions (4) and (6) are less restricting than the conditions of Chetayev's theorem [10] concerning Lyapunov instability. However, a weaker claim has been proved here – Krasovskii instability. The question of Lyapunov instability under conditions (4) and (6) remains open.

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